Letters

Comments on "Propagation Modes, Equivalent Circuits, and Characteristic Terminations for Multiconductor Transmission Lines with Inhomogeneous Dielectrics"

Y. Y. SUN

Abstract—**This letter intends to recognize the correct expression of the characteristic admittance matrix of multiconductor transmission lines.**

For uniformly coupled lossless $(n+1)$-conductor transmission lines, the voltages and currents are described by the equation pair

$$\frac{d}{dx} V(x,s) = -s[L]I(x,s) \qquad (1a)$$

$$\frac{d}{dx} I(x,s) = -s[C]V(x,s) \qquad (1b)$$

where V and I denote $n\times 1$ voltage and current vectors, respectively; $[L]$ and $[C]$ are $n\times n$ per-unit-length inductance and capacitance matrices, respectively. The distance x is measured along the line, and s is the Laplace transform variable.

Assume V and I both proportional to the same factor exp $(-jkx)$ and $s=j\omega, j=(-1)^{1/2}$, then (1a) and (1b) are reduced to

$$V = v[L]I \qquad (2a)$$

$$I = v[C]V \qquad (2b)$$

where $v=\omega/k$ is the velocity of wave propagation. Therefore

$$\frac{1}{v^2} V = [L][C]V \qquad (3a)$$

$$\frac{1}{v^2} I = [C][L]I. \qquad (3b)$$

In case the conductors are embedded in a transversely inhomogeneous dielectric medium, there always exist voltage and current modal matrices $[P]$ and $[Q]$ such that

$$[P]^{-1}[L][C][P] = [D] \qquad (4a)$$

$$[Q]^{-1}[C][L][Q] = [D] \qquad (4b)$$

where $[D]$ is a diagonal matrix, viz.,

$$[D] = \begin{bmatrix} v_1^{-2} & & & \bigcirc \\ & v_2^{-2} & & \\ & & \ddots & \\ \bigcirc & & & v_n^{-2} \end{bmatrix}$$

v_i is the ith mode wave propagation velocity. Taking transpose of these equations, one has

$$[P]^T[C][L]([P]^{-1})^T = [D] \qquad (5a)$$

$$[Q]^T[L][C]([Q]^{-1})^T = [D] \qquad (5b)$$

where $[C]$ and $[L]$ are assumed symmetric and the superscript T indicates transpose. Comparison of (4a) and (5b) as well as (4b) and (5a) will lead one to conclude that

$$[P]^T[Q] = [Q]^T[P] = [d] \qquad (6)$$

where $[d]$ is a diagonal matrix.

Marx[1] showed that the characteristic admittance matrix of the multiconductor transmission line is

$$[Y_0] = [Q]_n[Q]_n^T \qquad (7a)$$

$$[Y_0] = [Q]_n[P]_n^{-1} \qquad (7b)$$

where $[Q]_n$ and $[P]_n$ are such that their columns correspond to those of $[Q]$ and $[P]$ but normalized.

However, these expressions happen to be incorrect. To verify this assertion, consider the symmetrical case

$$[L] = \begin{bmatrix} L & M \\ M & L \end{bmatrix} \qquad (8a)$$

$$[C] = \begin{bmatrix} C & -K \\ -K & C \end{bmatrix}. \qquad (8b)$$

Now, it is readily shown that the normalized modal matrices which satisfy the conditions stated in Marx[1] are

$$[P]_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad (9a)$$

$$[Q]_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \qquad (9b)$$

So, from (7),

$$[Y_0] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (10)$$

This is incorrect. The correct expression should be

$$[Y_0] = \frac{1}{2} \begin{bmatrix} \sqrt{\frac{C-K}{L+M}} + \sqrt{\frac{C+K}{L-M}} & \sqrt{\frac{C-K}{L+M}} - \sqrt{\frac{C+K}{L-M}} \\ \sqrt{\frac{C-K}{L+M}} - \sqrt{\frac{C+K}{L-M}} & \sqrt{\frac{C-K}{L+M}} + \sqrt{\frac{C+K}{L-M}} \end{bmatrix}. \qquad (11)$$

This expression was obtained[1] through a groundless way by picking up

$$[P]_n = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad (12)$$

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The author is at P.O.Box 1-3 Lung-Tan (325), Taoyuan, Taiwan, China.

[1]K. D. Marx, "Propagation modes, equivalent circuits, and characteristic terminations for multiconductor transmission lines with inhomogeneous dielectrics," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 450–457, July 1973.

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$$[Q]_n = \begin{bmatrix} \sqrt{\frac{C-K}{L+M}} & \sqrt{\frac{C+K}{L-M}} \\ \sqrt{\frac{C-K}{L+M}} & -\sqrt{\frac{C+K}{L-M}} \end{bmatrix} \tag{13}$$

so as to fit (7b).

Actually, the characteristic admittance matrix in (11) is found via the well-known formulation [1]

$$[Y_0]=[Q]\,\mathrm{diag}\,[v][Q]^{-1}[C]=[Q]\,\mathrm{diag}\left[\frac{1}{v}\right][Q]^{-1}[L]^{-1} \tag{14a}$$

or

$$[Y_0]=[L]^{-1}[P]\,\mathrm{diag}\left[\frac{1}{v}\right][P]^{-1}=[C][P]\,\mathrm{diag}\,[v][P]^{-1}. \tag{14b}$$

The symmetry of (14) is proven in general in [1].

Equations (14) can also be derived in a slightly different manner. Treating (1a) and (1b) as if they were simple transmission line equations, one will have

$$[Y_0]=([C][L])^{-1/2}[C] \tag{15a}$$

or

$$[Y_0]=[L]^{-1}([L][C])^{1/2}. \tag{15b}$$

The equality of these two expressions is checked by employing the relation

$$[L]f([C][L])=f([L][C])[L] \tag{16}$$

where $f([C][L])$ is a function of $[C][L]$. Equation (16) is easily established as follows. Equation (4b) can be rewritten as

$$([L][Q])^{-1}[L][C]([L][Q])=[D]. \tag{17}$$

Let $f(\mu)$ be an arbitrary function of μ, then $f([C][L])$ and $[C][L]$ commute. So, they share the same modal matrix for similarity transformation [2]. That is, from (17)

$$([L][Q])^{-1}f([L][C])([L][Q])=f([D]). \tag{18}$$

Similarly, from (4b), one can have

$$[Q]^{-1}f([C][L])[Q]=f[D]. \tag{19}$$

Equating (18) and (19) yields (16).

The equivalence of (14) to (15) is seen by taking into account of

$$[Q][D]^{-1/2}[Q]^{-1}=[Q]\,\mathrm{diag}\,[v][Q]^{-1}=([C][L])^{-1/2} \tag{20a}$$

$$[P][D]^{1/2}[P]^{-1}=[P]\,\mathrm{diag}\left[\frac{1}{v}\right][P]^{-1}=([L][C])^{1/2}. \tag{20b}$$

In addition to the discrepancy indicated above, there are two more reasons to believe (14) or (15), rather than (7), correct.

1) In case the transmission line is a simple two-conductor structure, (14) justifies

$$Y_0=\left(\frac{C}{L}\right)^{1/2} \tag{21}$$

while (7) gives

$$Y_0=1. \tag{22}$$

2) For the simple transmission line, the energy transmission is equipartitioned in electric and magnetic energy densities [3]. That is

$$LI_f^2=CV_f^2. \tag{23}$$

A corresponding relation for multiconductor case is obtained by utilizing (14). That is

$$I_f^T[L]I_f=V_f^T[Y_0][L][Y_0]V_f=V_f^T[C]V_f. \tag{24}$$

The subscript f denotes one-direction, say forward, going wave.

Nonetheless, neither (7a) nor (7b) will give rise to (24).

References

[1] H. Amemiya, "Time domain analysis of multiple parallel transmission lines," *RCA Rev.*, vol. 28, pp. 241–276, 1967.

[2] J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* Oxford, England: Clarendon, 1965.

[3] V. Burke, R. J. Duffin, and D. Hazony, "Distortionless wave propagation in inhomogeneous media and transmission lines," *Quart. Appl. Math.*, pp. 183–194, July 1976.

Reply[2] by Kenneth D. Marx[3]

The above remarks by Y. Y. Sun regarding errors in our treatment[1] of the characteristic admittance matrix are simply incorrect. The difficulty seems to stem from a misunderstanding of the definition of orthogonality and normalization that is used[1]

$$\boldsymbol{I}_i\cdot \boldsymbol{V}_j=\delta_{ij} \tag{1}$$

where $\boldsymbol{I}_i$ is the ith current eigenvector, $\boldsymbol{V}_j$ is the jth voltage eigenvector, and δ_{ij} is the Kronecker delta. In the following, we reiterate the arguments which led to our results, and demonstrate that those results are indeed consistent and correct.

In Section II of our paper,[1] the following relations are derived for current and voltage vectors in a propagation mode:

$$\boldsymbol{V}=v\boldsymbol{L}\boldsymbol{I} \tag{2}$$

$$\boldsymbol{I}=v\boldsymbol{C}\boldsymbol{V}. \tag{3}$$

(The equation numbers have been changed from those of the original paper.[1]) The resulting eigenvalue equation for $\boldsymbol{V}$ is

$$(\boldsymbol{L}\boldsymbol{C})\boldsymbol{V}=\frac{1}{v^2}\boldsymbol{V}. \tag{4}$$

It is then pointed out that the current eigenvectors $\boldsymbol{I}_i$ are related through (3) to the voltage eigenvectors $\boldsymbol{V}_i$, which are solutions of (4). An equally valid alternative is to use (2) to obtain $\boldsymbol{I}_i$ from

$$\boldsymbol{I}_i=\frac{1}{v_i}\boldsymbol{L}^{-1}\boldsymbol{V}_i$$

where the propagation velocity v_i is the inverse square root of the eigenvalue $1/v_i^2$ in (4).

Through a straightforward argument, it is shown that current and voltage eigenvectors corresponding to different modes are orthogonal in the following sense:

$$\boldsymbol{I}_i\cdot \boldsymbol{V}_j=0 \tag{5}$$

unless the propagation velocities v_i and v_j are equal. In the

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degenerate case $v_i = v_j$, one can always orthogonalize through a generalization of the Gram–Schmidt procedure. Although not stated in the original paper,[1] a corollary to this result is that $\boldsymbol{I}_i \cdot \boldsymbol{I}_j$ and $\boldsymbol{V}_i \cdot \boldsymbol{V}_j$ are not zero in general for $i \neq j$. The reason for this is that the matrix $\boldsymbol{LC}$ is not symmetric, i.e., not Hermitian, or self-adjoint. Furthermore, one can use (2) and (3) and the fact that $\boldsymbol{L}$ and $\boldsymbol{C}$ are positive definite to show

$$\boldsymbol{I}_i \cdot \boldsymbol{V}_i \neq 0.$$

Hence, we are led to the definition of orthonormal sets of voltage and current eigenvector *pairs* as given in Section III-C of the original paper[1] and repeated in (1) above. This is an appropriate normalization for eigenvectors of an asymmetric matrix and its adjoint ($\boldsymbol{LC}$ and $\boldsymbol{CL}$). We note that then a mode with voltage $\boldsymbol{V}_i$ and current $\boldsymbol{I}_i$ has unit power (power $= \boldsymbol{V}_i \cdot \boldsymbol{I}_i = 1$).

Now, because of this definition of the normalization of eigenvectors, and because of the relationship (2) or (3) between the $\boldsymbol{I}_i$ and the $\boldsymbol{V}_i$, it is clear that there is a *constraint* on the eigenvectors that is *inconsistent* with the usual definition of normalization:

$$\boldsymbol{V}_i \cdot \boldsymbol{V}_i = 1 \tag{6}$$

$$\boldsymbol{I}_i \cdot \boldsymbol{I}_i = 1. \tag{7}$$

It is this definition that Sun uses to obtain his modal matrices

$$[P]_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$[Q]_n = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

in the three-conductor system that he proposes as a counterexample. It is not true, as he has stated, that "it is readily shown that [these are the] normalized modal matrices which satisfy the conditions stated[1]." It is not true because the eigenvectors in the original paper[1] satisfy (1)–(3), and not (6) and (7). The eigenvectors in $[P]_n$ and $[Q]_n$ do not, in general, satisfy the requisite equations (2) and (3). Hence, the supposition upon which Sun's comments are based is untrue.

Now consider the steps in the derivation of the formula for the characteristic admittance matrix $\boldsymbol{Y}_0$. Since there appears to be some confusion regarding this derivation, somewhat more detail will be provided than was given in the original paper.[1] For a line composed of $n+1$ conductors, (1) implies the linear independence in n-space of both sets of eigenvectors $\boldsymbol{V}_i, \boldsymbol{I}_i$, $i = 1, 2, \cdots, n$. Let $\boldsymbol{M}_V$ and $\boldsymbol{M}_I$ be the matrices whose columns are voltage and current eigenvectors normalized according to (1). Hence

$$\boldsymbol{M}_V \boldsymbol{M}_I^T = \boldsymbol{M}_I \boldsymbol{M}_V^T = \boldsymbol{U}$$

where $\boldsymbol{U}$ is the identity matrix. This means that

$$\boldsymbol{M}_V^{-1} = \boldsymbol{M}_I^T.$$

Then it can be seen that it is possible to express an arbitrary vector $\boldsymbol{E}$ in the form

$$\boldsymbol{E} = \boldsymbol{M}_V \boldsymbol{A}$$

$$= \sum_i A_i \boldsymbol{V}_i$$

where $\boldsymbol{A}$, the vector whose components are the eigenvector expansion coefficients, is given by

$$\boldsymbol{A} = \boldsymbol{M}_I^T \boldsymbol{E}$$

i.e.,

$$A_i = \boldsymbol{I}_i \cdot \boldsymbol{E}.$$

Now, consider a wave traveling in the forward direction, which we take to be the z-direction. Let this wave be given by $\boldsymbol{V}_f(z,t)$, a function of space and time. If

$$\boldsymbol{A} = \boldsymbol{M}_I^T \boldsymbol{V}_f$$

then

$$\boldsymbol{V}_f = \sum_i A_i \boldsymbol{V}_i \tag{8}$$

at any point in space and time. The current in this forward wave is

$$\boldsymbol{I}_f = \sum_i A_i \boldsymbol{I}_f. \tag{9}$$

The reason that (9) follows from (8) can be simply stated as follows: when the voltage vector is given as the superposition of voltage eigenvectors as in (8), and the current vector is written as the superposition of the corresponding current eigenvectors, the coefficients in the two expansions must be the same, in order that the telegrapher's equations

$$\frac{\partial \boldsymbol{V}}{\partial z} = -\boldsymbol{L} \frac{\partial \boldsymbol{I}}{\partial t} \tag{10}$$

$$\frac{\partial \boldsymbol{I}}{\partial z} = -\boldsymbol{C} \frac{\partial \boldsymbol{V}}{\partial t} \tag{11}$$

are satisfied. Since the solution to (10) and (11) must be unique, this is the only result possible.

A more detailed explanation of this same point: let us rewrite (8), showing the functional dependences of $\boldsymbol{V}_f$ and A_i on z and t

$$\boldsymbol{V}_f(z,t) = \sum_i A_i(z - v_i t) \boldsymbol{V}_i \tag{8$'$}$$

where $A_i(z - v_i t)$ is the ith component of

$$\boldsymbol{A} = \boldsymbol{M}_I^T \boldsymbol{V}_f(z,t).$$

The components of the vector $\boldsymbol{A}$ can be evaluated as functions of their respective arguments at any time t by sampling along the entire (infinite) line, or at any position z by sampling over all time. The reason that the components must have the functional form $A_i(z - v_i t)$ is that in this way $\boldsymbol{V}_f(z,t)$ consists of a forward wave satisfying the equation

$$\frac{\partial^2 \boldsymbol{V}}{\partial z^2} = \boldsymbol{LC} \frac{\partial^2 \boldsymbol{V}}{\partial t^2}$$

obtained when (10) and (11) are combined. Since the solution to this problem must be unique, the representation

$$\boldsymbol{V}_f(z,t) = \sum_i A_i(z - v_i t) \boldsymbol{V}_i$$

must be unique. Now if the voltage is given by this expression, we can use either (10) or (11) to evaluate the current. We will use (11). Then

$$\frac{\partial \boldsymbol{I}_f(z,t)}{\partial z} = -\boldsymbol{C} \frac{\partial}{\partial t} \sum_i A_i(z - v_i t) \boldsymbol{V}_i$$

$$= -\sum_i -v_i A_i'(z - v_i t) \boldsymbol{C} \boldsymbol{V}_i$$

$$= \sum_i A_i'(z - v_i t) \boldsymbol{I}_i$$

where we have used (3) to eliminate $v_i \boldsymbol{C} \boldsymbol{V}_i$, and A_i' represents differentiation of A_i with respect to its single argument. Integrating with respect to z, we obtain

$$I_f(z,t)=\sum_i A_i(z-v_it)I_i$$

or

$$I_f=M_IA$$

as claimed in the original paper.[1] Since

$$A=M_I^TV_f$$

we have

$$I_f=M_IM_I^TV_f$$

which shows that the correct definition of the admittance matrix is

$$\bar{Y}_0=M_IM_I^T. \tag{12a}$$

((8a) in the original paper[1]). This is true only if the eigenvectors are properly normalized.

Suppose that the eigenvectors are *not normalized.* We can still write

$$V_f=\sum_i A_iV_i$$

where

$$A=M_V^{-1}V_F.$$

In this case we no longer have

$$M_V^{-1}=M_I^T.$$

But precisely the same sort of argument as that just preceding results in

$$Y_0=M_IM_V^{-1} \tag{12b}$$

((8b) in the original paper[1]) *as long as the eigenvectors satisfy* (2) *and* (3).

In the original paper[1] we used as an example a three-conductor line with symmetric conductors, but inhomogeneous dielectric. In this case, the inductance and capacitance matrices have the form given by Sun's (8a) and (8b). We derived his formula (11) for the characteristic admittance matrix by a direct application of (12b) above, using appropriate unnormalized eigenvectors, which is completely justified as noted above and explicitly stated.[1] Hence, Sun is wrong in asserting that our approach is groundless.

Specifically, the eigenvector pairs used in that case were

$$V_1=\begin{pmatrix}1\\1\end{pmatrix},\quad I_1=\frac{1}{Z_e}\begin{pmatrix}1\\1\end{pmatrix}$$

and

$$V_2=\begin{pmatrix}1\\-1\end{pmatrix},\quad I_2=\frac{1}{Z_o}\begin{pmatrix}1\\-1\end{pmatrix}$$

where the even- and odd-mode characteristic impedances are

$$Z_e=\sqrt{\frac{L+M}{C-K}},\quad Z_o=\sqrt{\frac{L-M}{C+K}}.$$

(We have used Sun's notation for the components of $\bar{L}$ and $\bar{C}$.) Using the even- and odd-mode propagation velocities

$$V_1=\frac{1}{\sqrt{(L+M)(C-K)}},\quad V_2=\frac{1}{\sqrt{(L-M)(C+K)}}$$

it is easily seen that these eigenvectors satisfy (2) and (3). Then

$$M_V=\begin{pmatrix}1&1\\1&-1\end{pmatrix},\quad M_I=\begin{bmatrix}\frac{1}{Z_e}&\frac{1}{Z_o}\\\frac{1}{Z_e}&\frac{-1}{Z_o}\end{bmatrix}$$

and (12b) yields

$$Y_0=\frac{1}{2}\begin{bmatrix}\frac{1}{Z_e}+\frac{1}{Z_o}&\frac{1}{Z_e}-\frac{1}{Z_o}\\\frac{1}{Z_e}-\frac{1}{Z_o}&\frac{1}{Z_e}+\frac{1}{Z_o}\end{bmatrix} \tag{13}$$

which is identical to Sun's (11). To compare this with (12a) above, note that the *normalized* eigenvector pairs are obtained by multiplying the unnormalized ones by $\sqrt{Z\binom{e}{o}/2}$, so that

$$V_1=\sqrt{\frac{Z_e}{2}}\begin{pmatrix}1\\1\end{pmatrix},\quad I_1=\sqrt{\frac{1}{2Z_e}}\begin{pmatrix}1\\1\end{pmatrix}$$

and

$$V_2=\sqrt{\frac{Z_o}{2}}\begin{pmatrix}1\\-1\end{pmatrix},\quad I_2=\sqrt{\frac{1}{2Z_o}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

This solution for the normalized eigenvectors is unique. This clearly points out the constraint on the eigenvectors referred to in connection with (6) and (7). Then

$$M_I=\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{Z_e}}&\frac{1}{\sqrt{Z_o}}\\\frac{1}{\sqrt{Z_e}}&\frac{-1}{\sqrt{Z_o}}\end{bmatrix},\quad M_I^T=\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{Z_e}}&\frac{1}{\sqrt{Z_e}}\\\frac{1}{\sqrt{Z_o}}&\frac{-1}{\sqrt{Z_o}}\end{bmatrix}$$

and (12a) yields the identical result (13) for Y_0.

As further proposed counterexamples, Sun claims that our formulas yield incorrect results for the characteristic admittance of a simple two-conductor line and for equipartition of energy in it. But from (1)–(3) one sees that the normalized eigenvectors for the simple line are

$$V=\left(\frac{L}{C}\right)^{1/4},\quad I=\left(\frac{C}{L}\right)^{1/4}.$$

(Recall that the propagation velocity is $v=1/\sqrt{LC}$.) Again, this solution for the eigenvectors is *unique*. Hence, (12a) yields

$$Y_0=I/V=\sqrt{\frac{C}{L}}$$

as required. Furthermore,

$$LI^2=\sqrt{LC}=CV^2$$

so correct equipartition of energy is obtained for the normalized eigenvectors. Since voltage and current in any wave is represented by a single constant times the respective eigenvectors, it is clear that equipartition is obtained for any wave.

These results all indicate that Sun's objections are simply based on a misunderstanding of our definition of normalization. There is therefore no reason to question the results in the original paper.[1]